Outline

- Introduction
- Griffith’s energy criterion
- Elastic energy release
- Crack growth resistance
- Crack tip stress fields
- Variational formulation of energy balance
- Phase field numerical implementation
- Examples
Introduction

Max-Planck-Institut für Eisenforschung GmbH.
Introduction

Cu internal crack formed by linking voids
Objectives of Fracture Mechanics

- What is the relationship between material strength and crack size?
Brittle vs Ductile Fracture

- Brittle fracture: No apparent plastic deformation before fracture, unstable crack propagation
- Ductile fracture: Extensive plastic deformation before fracture, stable crack propagation
Historical Developments: Inglis, 1913

For a crack,

\[ \sigma_3 = \sigma_1 + 2 \frac{b}{a} \]

\[ \sigma_3 \rightarrow \infty \quad \text{as} \quad a \rightarrow 0 \]
Historical Developments: Griffith, 1921

- Experiments on fracture strength of glass fibers
- Fracture strength increases as fiber diameter decreases

**TABLE 1.1. Strength of glass fibers according to Griffith’s experiments.**

<table>
<thead>
<tr>
<th>Diameter (10^{-3} in)</th>
<th>Breaking stress (lb/in^{2})</th>
<th>Diameter (10^{-3} in)</th>
<th>Breaking stress (lb/in^{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>40.00</td>
<td>24 900</td>
<td>0.95</td>
<td>117 000</td>
</tr>
<tr>
<td>4.20</td>
<td>42 300</td>
<td>0.75</td>
<td>134 000</td>
</tr>
<tr>
<td>2.78</td>
<td>50 800</td>
<td>0.70</td>
<td>164 000</td>
</tr>
<tr>
<td>2.25</td>
<td>64 100</td>
<td>0.60</td>
<td>185 000</td>
</tr>
<tr>
<td>2.00</td>
<td>79 600</td>
<td>0.56</td>
<td>154 000</td>
</tr>
<tr>
<td>1.85</td>
<td>88 500</td>
<td>0.50</td>
<td>195 000</td>
</tr>
<tr>
<td>1.75</td>
<td>82 600</td>
<td>0.38</td>
<td>232 000</td>
</tr>
<tr>
<td>1.40</td>
<td>85 200</td>
<td>0.26</td>
<td>332 000</td>
</tr>
<tr>
<td>1.32</td>
<td>99 500</td>
<td>0.165</td>
<td>498 000</td>
</tr>
<tr>
<td>1.15</td>
<td>88 700</td>
<td>0.130</td>
<td>491 000</td>
</tr>
</tbody>
</table>
Glass fibers with artificial cracks reveal a scaling of fracture strength with crack size.

Historical Developments: Griffith, 1921

<table>
<thead>
<tr>
<th>Crack Length, 2a (mm)</th>
<th>Measured Strength, $\sigma_c$ (MPa)</th>
<th>$\sigma_c \sqrt{a}$ (MPa$\sqrt{m}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>sample 1</td>
<td>3.8</td>
<td>6.0</td>
</tr>
<tr>
<td>sample 2</td>
<td>6.9</td>
<td>4.3</td>
</tr>
<tr>
<td>sample 3</td>
<td>13.7</td>
<td>3.3</td>
</tr>
<tr>
<td>sample 4</td>
<td>22.6</td>
<td>2.5</td>
</tr>
</tbody>
</table>

(Data from the Griffith experiment)

$c\sqrt{a} = \text{const}$
Griffith’s Energy Balance

- All energetic changes are caused by changes in crack size:

\[ \frac{W}{a} = \frac{U_e}{a} + \frac{U_i}{a} + \frac{U_k}{a} + \frac{U}{a} \]

- For brittle materials and slow processes:

\[ W = U_e \quad \frac{a}{a} = \frac{U}{a} = 2_s \]

\( s \): Energy required to form a unit surface area
Griffith’s Energy Balance

- Using Inglis’s solution for an elliptical crack:

\[ W = a^2 B \frac{a^2}{E} \quad U = 4aB_s \]

\[ \frac{a^2}{a} = 2 \quad a^2 B \frac{2}{E} \quad \frac{U}{a} = 4B_s \]

- From energy balance:

\[ c = \sqrt{\frac{2E_s}{a}} \]
• Irwin, Orowan (1948):

\[ c = \sqrt{\frac{2E(\frac{s}{a} + \frac{p}{a})}{a}} \]

\( s \): Plastic work per unit surface area created

• Typically in metals, \( p \approx 1000 \)

• Not a material constant
\[ G = \frac{-\partial P}{\partial a} \]

\[ G: \text{Energy released during fracture per created crack surface area} \]

- Energy release rate failure criterion,

\[ G \quad G_c = 2\left( s + p \right) \]
Energy Release Rate Measurement

Fixed grips

$$G = \frac{1}{B} \frac{(OAB)}{\Delta a}$$

Dead loads

$$\Pi = U_e - W$$

OAB = ABCD - (OBD - OAC)

B: thickness
\[ G = \frac{1}{B} \frac{\text{shaded area}}{a_4 - a_3} \]
• Energy balance with plasticity:

\[
\frac{\partial U}{\partial a} = \frac{U}{a} + \frac{U_i}{a}
\]

\[
R = \frac{U}{a} + \frac{U_i}{a}
\]

\( R \): Crack growth resistance

• R increases with growing crack size in plastic materials

• Not a material constant
• Flat R-Curve: Brittle materials

\[
\frac{\partial G}{\partial a} \leq \frac{\partial R}{\partial a} \rightarrow \text{Stable crack growth}
\]
R-Curves: Ductile

- Rising R-Curve: Ductile materials
Crack Modes

Mode I: Opening

Mode II: In-plane shear

Mode III: Out-of-plane shear
Crack Tip Stress Field: Mode I

Westergaard (1937)

\[ xx = \frac{K_I}{\sqrt{2}} \frac{\cos \frac{2}{2} 1 \sin \frac{2}{2} \cos \frac{3}{2}}{r} \]

\[ yy = \frac{K_I}{\sqrt{2}} \frac{\cos \frac{2}{2} 1 + \sin \frac{2}{2} \cos \frac{3}{2}}{r} \]

\[ xy = \frac{K_I}{\sqrt{2}} \frac{\cos \frac{2}{2} \sin \frac{2}{2} \cos \frac{3}{2}}{r} \]
Crack Tip Stress Field: Mode II

\[
xx = \frac{K_{II}}{\sqrt{2} r} \sin \frac{2}{2} \left( 2 + \cos \frac{3}{2} \right)
\]

\[
yy = \frac{K_{II}}{\sqrt{2} r} \cos \frac{3}{2} \sin \frac{2}{2}
\]

\[
xy = \frac{K_{II}}{\sqrt{2} r} \cos \frac{3}{2} \sin \frac{2}{2} \cos \frac{3}{2}
\]

\[K_I, K_{II} : \text{Stress intensity factor}\]
K-G relationship

- Work to required to open a crack, $G$, is the same as the work required to close a crack.

\[
W = \int_0^a y u_y \, dx
\]

\[
u_y = \left( +1 \right) K_1 \left( a + \frac{a}{2} \right) \sqrt{\frac{a}{2}} x
\]

\[
y y = \frac{K_1 (a)}{\sqrt{2}} \frac{1}{x}
\]
**K-G relationship**

**Mode I**

\[
G_I = \begin{cases} 
\frac{K_I^2}{E} & \text{plane stress} \\
(1 - \nu^2) \frac{K_I^2}{E} & \text{plane strain}
\end{cases}
\]

**Mixed mode**

\[
G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}
\]

\[
E' = \begin{cases} 
\frac{E}{1 - \nu^2} & \text{for plane strain} \\
\frac{E}{E} & \text{for plane stress}
\end{cases}
\]
• Formulate Griffith’s energy balance as a minimum energy principle

\[
\frac{\partial}{\partial a} + \frac{\partial U}{\partial a} = 0 \rightarrow \min_a \left( P + U \right)
\]

• Couple with mechanics

\[
\min_{u,a} \int (u, a) d\gamma + \int_a 2 \sigma da
\]
Phase Field Regularization

• Minimization over all possible crack surfaces is numerically challenging

• Phase Field approximation of the surface integral

\[ \min_{u,\varphi} \int_{\Omega} \varphi^2 \Pi(u) d\Omega + \int_{\Omega} 2 \left( \gamma_s \ell |\nabla \varphi|^2 + \frac{\gamma_s}{\ell} (1 - \varphi)^2 \right) d\Omega \]

\[ \text{Elastic energy} \quad \text{Surface energy} \]

• Starionary condition:

\[ \nabla^2 \frac{\varphi}{\nabla u} = 0 \quad 2 \frac{s}{\ell} \frac{s}{\ell} \left(1\right) \quad 2 \quad = 0 \]
Examples

- PolyCrystalline fracture mechanics

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{11}$</td>
<td>GPa</td>
<td>168.0</td>
</tr>
<tr>
<td>$C_{12}$</td>
<td>GPa</td>
<td>121.4</td>
</tr>
<tr>
<td>$C_{44}$</td>
<td>GPa</td>
<td>28.34</td>
</tr>
<tr>
<td>$\dot{\gamma}_0$</td>
<td>s$^{-1}$</td>
<td>1e-3</td>
</tr>
<tr>
<td>$n$</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>$g_0$</td>
<td>MPa</td>
<td>31</td>
</tr>
<tr>
<td>$g_\infty$</td>
<td>MPa</td>
<td>63</td>
</tr>
<tr>
<td>$a$</td>
<td></td>
<td>2.25</td>
</tr>
<tr>
<td>$h_0$</td>
<td>MPa</td>
<td>75</td>
</tr>
<tr>
<td>coplanar $h_{\alpha\beta}$</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>non-coplanar $h_{\alpha\beta}$</td>
<td></td>
<td>1.4</td>
</tr>
<tr>
<td>$G_0$</td>
<td>Jm$^{-2}$</td>
<td>1.0</td>
</tr>
<tr>
<td>$l_0$</td>
<td>$\mu$m</td>
<td>1.5</td>
</tr>
<tr>
<td>$M$</td>
<td>s$^{-1}$</td>
<td>0.01</td>
</tr>
</tbody>
</table>
• Evolving crack patterns
Examples

- Elastic energy release and crack tip stress
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damask.mpie.de