• Introduction
• Griffith’s energy criterion
• Elastic energy release
• Crack growth resistance
• Crack tip stress fields
• Variational formulation of energy balance
• Phase field numerical implementation
• Examples
Introduction
Introduction

Cu internal crack formed by linking voids

steel
• What is the relationship between material strength and crack size?
Brittle vs Ductile Fracture

- Brittle fracture: No apparent plastic deformation before fracture, unstable crack propagation
- Ductile fracture: Extensive plastic deformation before fracture, stable crack propagation
For a crack,

\[ \sigma_3 = \sigma_1 \left( 1 + 2 \frac{b}{a} \right) \]

\[ a \to 0 \Rightarrow \sigma_3 \to \infty \]
Historical Developments: Griffith, 1921

- Experiments on fracture strength of glass fibers
- Fracture strength increases as fiber diameter decreases

<table>
<thead>
<tr>
<th>Diameter (10^-3 in)</th>
<th>Breaking stress (lb/in²)</th>
<th>Diameter (10^-3 in)</th>
<th>Breaking stress (lb/in²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>40.00</td>
<td>24900</td>
<td>0.95</td>
<td>117000</td>
</tr>
<tr>
<td>4.20</td>
<td>42300</td>
<td>0.75</td>
<td>134000</td>
</tr>
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<td>2.78</td>
<td>50800</td>
<td>0.70</td>
<td>164000</td>
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<td>2.25</td>
<td>64100</td>
<td>0.60</td>
<td>185000</td>
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<td>2.00</td>
<td>79600</td>
<td>0.56</td>
<td>154000</td>
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<tr>
<td>1.85</td>
<td>88500</td>
<td>0.50</td>
<td>195000</td>
</tr>
<tr>
<td>1.75</td>
<td>82600</td>
<td>0.38</td>
<td>232000</td>
</tr>
<tr>
<td>1.40</td>
<td>85200</td>
<td>0.26</td>
<td>332000</td>
</tr>
<tr>
<td>1.32</td>
<td>99500</td>
<td>0.165</td>
<td>498000</td>
</tr>
<tr>
<td>1.15</td>
<td>88700</td>
<td>0.130</td>
<td>491000</td>
</tr>
</tbody>
</table>
• Glass fibers with artificial cracks reveal a scaling of fracture strength with crack size

\[ \sigma_c \sqrt{a} = \text{const} \]

<table>
<thead>
<tr>
<th>(Data from the Griffith experiment)</th>
<th>Crack Length, 2a mm</th>
<th>Measured Strength, ( \sigma_c ) MPa</th>
<th>( \sigma_c \sqrt{a} ) MPa(\sqrt{m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sample 1</td>
<td>3.8</td>
<td>6.0</td>
<td>0.26</td>
</tr>
<tr>
<td>sample 2</td>
<td>6.9</td>
<td>4.3</td>
<td>0.25</td>
</tr>
<tr>
<td>sample 3</td>
<td>13.7</td>
<td>3.3</td>
<td>0.27</td>
</tr>
<tr>
<td>sample 4</td>
<td>22.6</td>
<td>2.5</td>
<td>0.27</td>
</tr>
</tbody>
</table>
Griffith’s Energy Balance

- All energetic changes are caused by changes in crack size:

\[
\frac{\partial W}{\partial a} = \frac{\partial U_e}{\partial a} + \frac{\partial U_i}{\partial a} + \frac{\partial U_k}{\partial a} + \frac{\partial U_\Gamma}{\partial a}
\]

- For brittle materials and slow processes:

\[
\Pi = U_e - W \implies -\frac{\partial \Pi}{\partial a} = \frac{\partial U_\Gamma}{\partial a} = 2\gamma_s
\]

\(\gamma_s\) : Energy required to form a unit surface area
Griffith’s Energy Balance

• Using Inglis’s solution for an elliptical crack:

\[ W = \pi a^2 B \frac{\sigma^2}{E} \]

\[ U_\Gamma = 4aB\gamma_s \]

\[ \Rightarrow - \frac{\partial \Pi}{\partial a} = 2\pi a B \frac{\sigma^2}{E} \quad \frac{\partial U_\Gamma}{\partial a} = 4B\gamma_s \]

• From energy balance:

\[ \sigma_c = \sqrt{\frac{2E\gamma_s}{\pi a}} \]
Energy Balance for Ductile Materials

• Irwin, Orowan (1948):

\[ \sigma_c = \sqrt{\frac{2E(\gamma_s + \gamma_p)}{\pi a}} \]

\( \gamma_s \) : Plastic work per unit surface area created

• Typically in metals, \( \gamma_p \approx 1000\gamma_s \)

• Not a material constant
Energy Release Rate: Irwin (1956)

\[ G = -\frac{\partial \Pi}{\partial a} \]

\( G \): Energy released during fracture per created crack surface area

- Energy release rate failure criterion,

\[ G \geq G_c = 2\left(\gamma_s + \gamma_p\right) \]
Energy Release Rate Measurement

Fixed grips

Dead loads

\[ G = \frac{1}{B} \frac{(OAB)}{\Delta a} \]

\[ \Pi = U_e - W \]

OAB = ABCD - (OBD - OAC)
Energy Release Rate Measurement

\[ G = \frac{1}{B} \frac{\text{shaded area}}{a_4 - a_3} \]
Energy balance with plasticity:

\[-\frac{\partial \Pi}{\partial a} = \frac{\partial U_\Gamma}{\partial a} + \frac{\partial U_i}{\partial a}\]

\[R \equiv \frac{\partial U_\Gamma}{\partial a} + \frac{\partial U_i}{\partial a}\]

R increases with growing crack size in plastic materials

Not a material constant
- Flat R-Curve: Brittle materials

\[
\frac{\partial G}{\partial a} \leq \frac{\partial R}{\partial a} \rightarrow \text{Stable crack growth}
\]
• Rising R-Curve: Ductile materials
Crack Modes

Mode I: Opening

Mode II: In-plane shear

Mode III: Out-of-plane shear
• Westergaard (1937)

\[
\begin{align*}
\sigma_{xx} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \\
\sigma_{yy} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \\
\tau_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2}
\end{align*}
\]
Crack Tip Stress Field: Mode II

\[
\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin \theta \left( 2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right)
\]

\[
\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2}
\]

\[
\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \right)
\]

\[K_I, K_{II} : \text{Stress intensity factor}\]
• Work to required to open a crack, G, is the same as the work required to close a crack.

\[
\Delta W = \int_0^{\Delta a} \sigma_{yy} u_y \, dx
\]

\[
u_y = \frac{(\kappa + 1) K_I (a + \Delta a)}{2 \mu} \sqrt{\frac{\Delta a - x}{2\pi}}
\]

\[
\sigma_{yy} = \frac{K_I (a)}{\sqrt{2\pi x}}
\]
K-G relationship

Mode I

\[
G_I = \begin{cases} 
\frac{K_I^2}{E} & \text{plane stress} \\
(1 - \nu^2) \frac{K_I^2}{E} & \text{plane strain}
\end{cases}
\]

Mixed mode

\[
G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}
\]

\[
E' = \begin{cases} 
\frac{E}{1 - \nu^2} & \text{for plane strain} \\
\frac{E}{E} & \text{for plane stress}
\end{cases}
\]
• Formulate Griffith’s energy balance as a minimum energy principle

\[
\frac{\partial \Pi}{\partial a} + \frac{\partial U_\Gamma}{\partial a} = 0 \rightarrow \min_a \left( \Pi + U_\Gamma \right)
\]

• Couple with mechanics

\[
\min_{u,a} \int_{\Omega} \Pi(u,a) \, d\Omega + \int_a 2\gamma_s \, da
\]
Phase Field Regularization

- Minimization over all possible crack surfaces is numerically challenging

- Phase Field approximation of the surface integral

\[
\min_{u,\varphi} \int_{\Omega} \varphi^2 \Pi(u) d\Omega + \int_{\Omega} 2 \left( \gamma_s \ell |\nabla \varphi|^2 + \frac{\gamma_s}{\ell} (1 - \varphi)^2 \right) d\Omega
\]

- Stationary condition:

\[
\nabla \cdot \varphi^2 \frac{\delta \Pi}{\delta \nabla u} = 0 \quad 2 \gamma_s \ell \Delta \varphi - \frac{\gamma_s}{\ell} (1 - \varphi) - 2 \varphi \Pi = 0
\]
Examples

- PolyCrystalline fracture mechanics

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{11}$</td>
<td>GPa</td>
<td>168.0</td>
</tr>
<tr>
<td>$C_{12}$</td>
<td>GPa</td>
<td>121.4</td>
</tr>
<tr>
<td>$C_{44}$</td>
<td>GPa</td>
<td>28.34</td>
</tr>
<tr>
<td>$\dot{\gamma}_0$</td>
<td>s$^{-1}$</td>
<td>1e-3</td>
</tr>
<tr>
<td>$n$</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>$g_0$</td>
<td>MPa</td>
<td>31</td>
</tr>
<tr>
<td>$g_\infty$</td>
<td>MPa</td>
<td>63</td>
</tr>
<tr>
<td>$a$</td>
<td></td>
<td>2.25</td>
</tr>
<tr>
<td>$h_0$</td>
<td>MPa</td>
<td>75</td>
</tr>
<tr>
<td>coplanar $h_{\alpha\beta}$</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>non-coplanar $h_{\alpha\beta}$</td>
<td></td>
<td>1.4</td>
</tr>
<tr>
<td>$G_0$</td>
<td>Jm$^{-2}$</td>
<td>1.0</td>
</tr>
<tr>
<td>$l_0$</td>
<td>$\mu$m</td>
<td>1.5</td>
</tr>
<tr>
<td>M</td>
<td>s$^{-1}$</td>
<td>0.01</td>
</tr>
</tbody>
</table>
Examples

- Evolving crack patterns
• Elastic energy release and crack tip stress
Summary

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damask.mpie.de