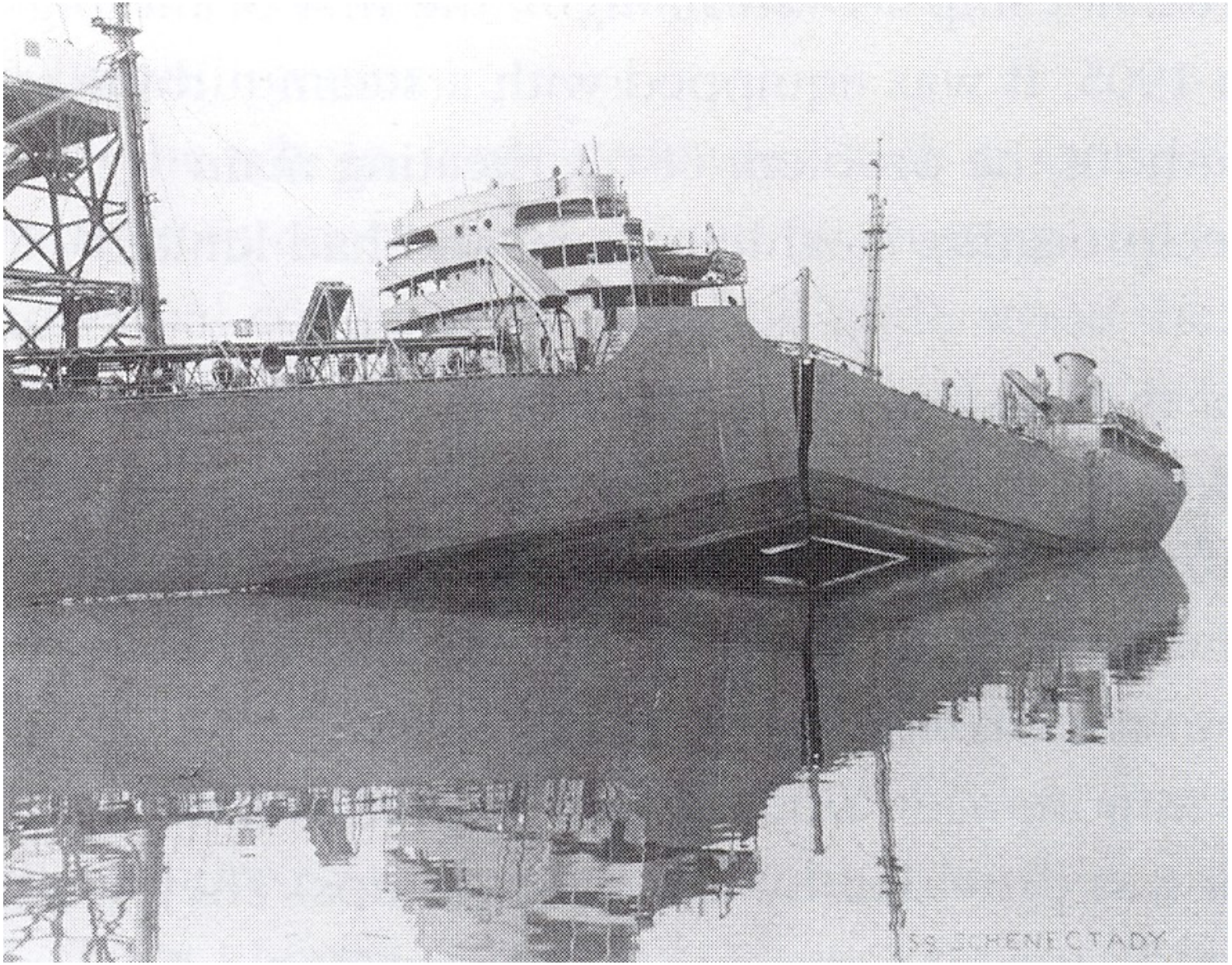
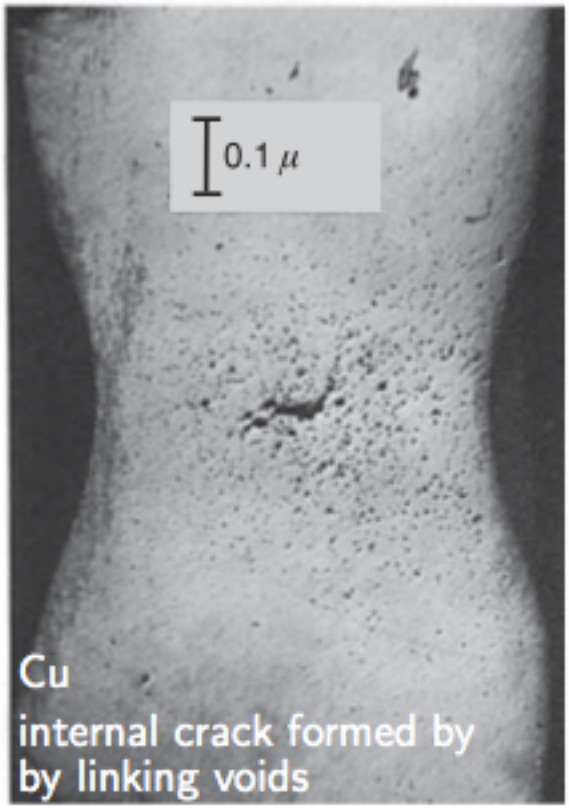
Outline

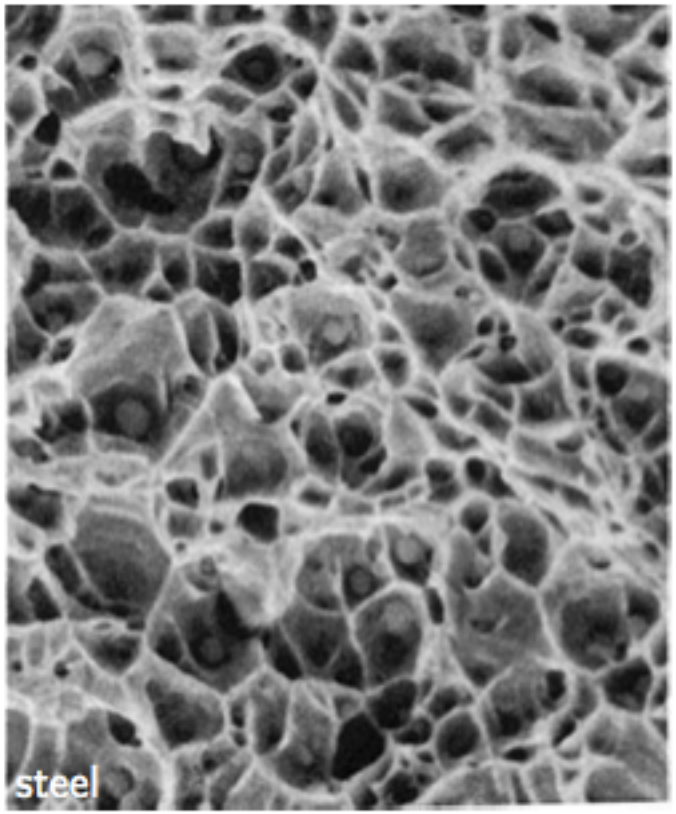
- Indroduction
- Griffith's energy criterion
- Elastic energy release
- Crack growth resistance
- Crack tip stress fields
- Variational formulation of energy balance
- Phase field numerical implementation
- Examples

Introduction

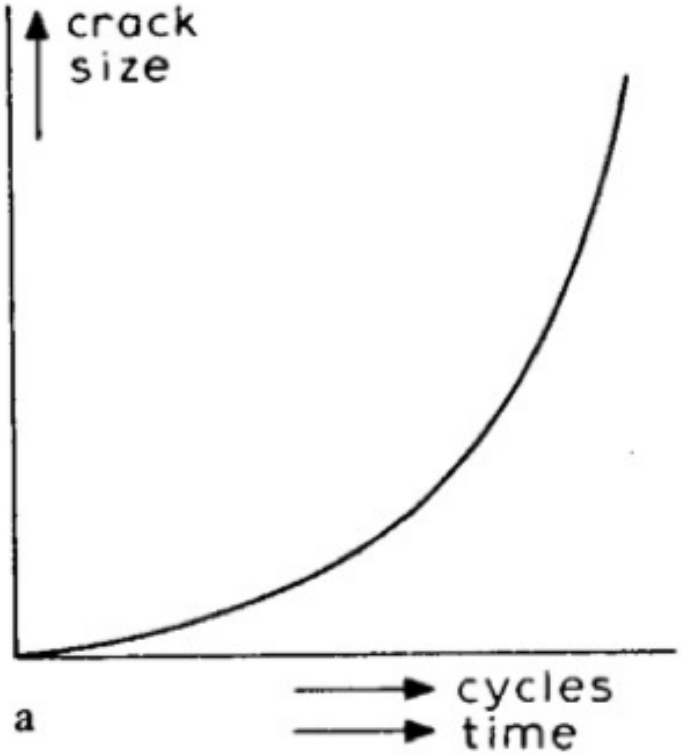


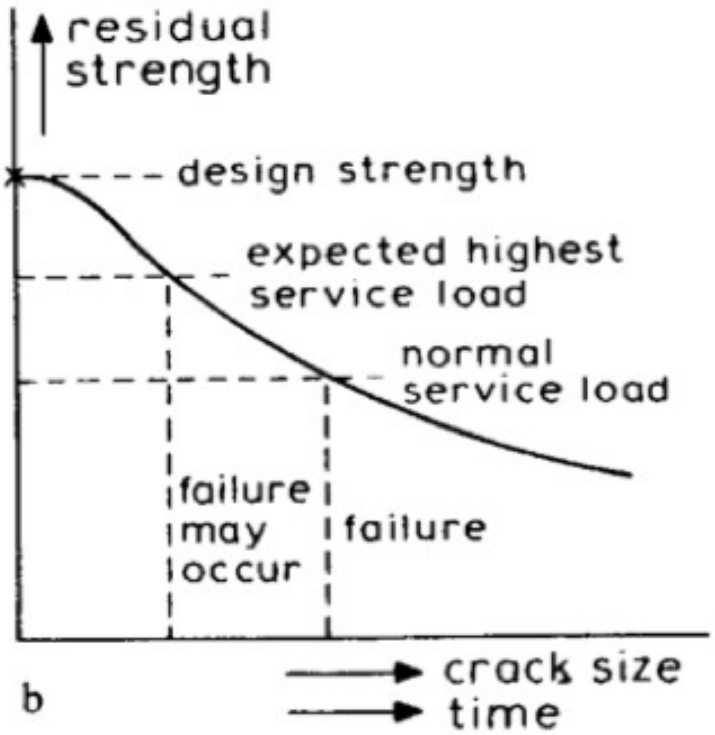
Introduction





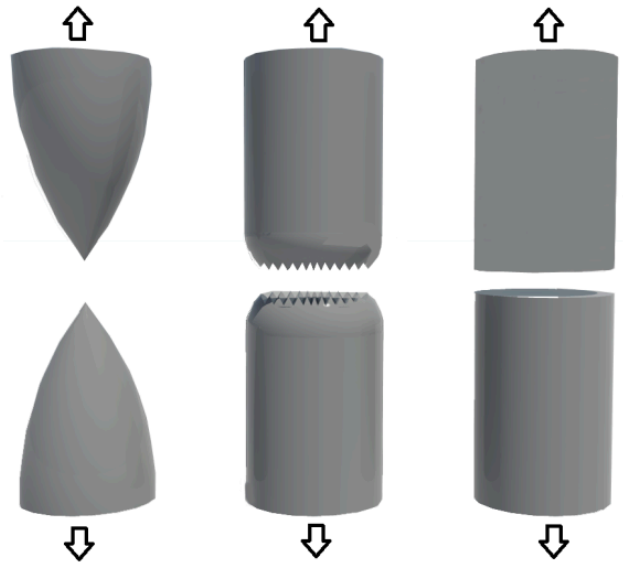
Objectives of Fracture Mechanics

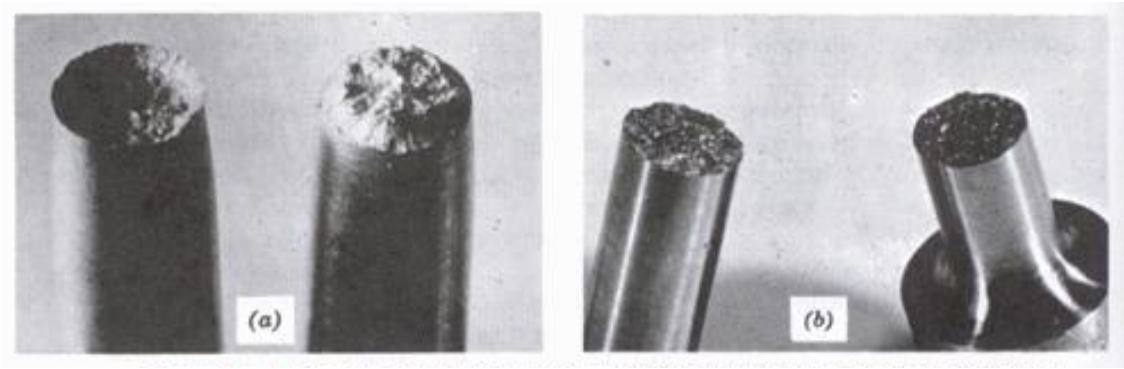




- What is the relationship between material strength and crack size?

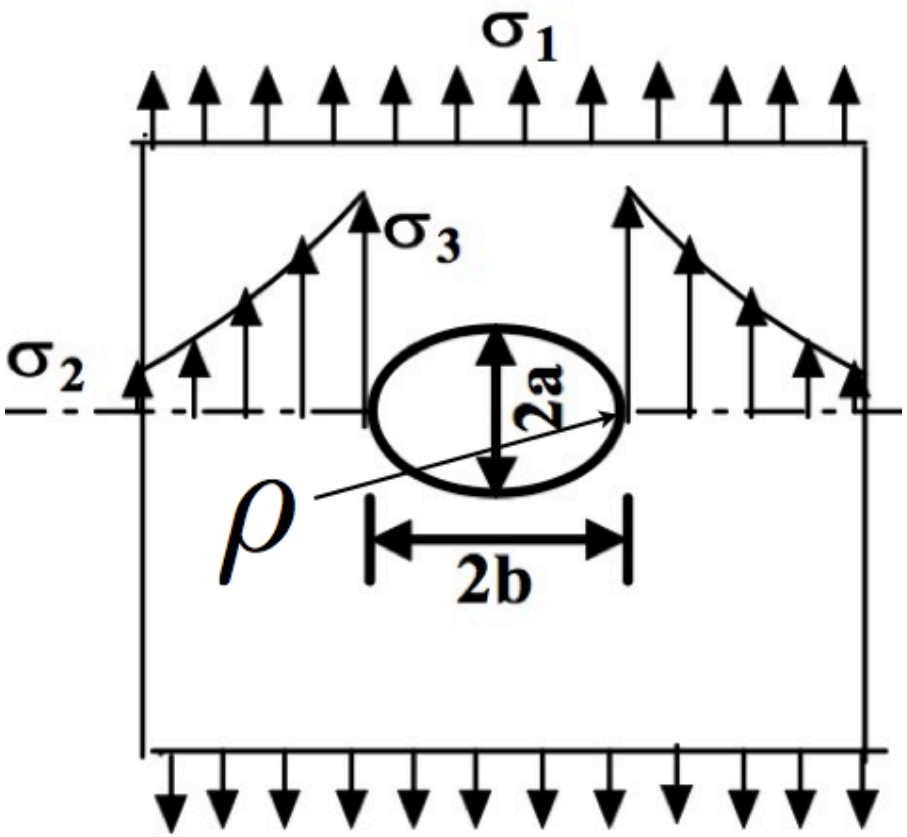
Brittle vs Ductile Fracture





- Brittle fracture: No apparent plastic deformation before fracture, unstable crack propagation
- Ductile fracture: Extensive plastic deformation before fracture, stable crack propagation

Historical Developments: Inglis, 1913



$$\sigma_3 = \sigma_1 \left(1 + 2\frac{b}{a} \right)$$

For a crack,

$$a \to 0 \Rightarrow \sigma_3 \to \infty$$

Historical Developments: Griffith, 1921

- Experiments on fracture strength of glass fibers
- Fracture strength increases as fiber diameter decreases

TABLE 1.1. Strength of glass fibers according to Griffith's experiments.

Diameter (10^{-3} in)	Breaking stress (lb/in^2)	Diameter (10^{-3} in)	Breaking stress (lb/in^2)
40.00	24 900	0.95	117 000
4.20	42 300	0.75	134 000
2.78	50 800	0.70	164 000
2.25	64 100	0.60	185 000
2.00	79 600	0.56	154 000
1.85	88 500	0.50	195 000
1.75	82 600	0.38	232 000
1.40	85 200	0.26	332 000
1.32	99 500	0.165	498 000
1.15	88 700	0.130	491 000

Historical Developments: Griffith, 1921

- Glass fibers with artificial cracks reveal a scaling of fracture strength with crack size

	Crack Length, $2a$ mm	Measured Strength, σ_c MPa	$\sigma_c\sqrt{a}$ MPa$\sqrt{\text{m}}$
sample 1	3.8	6.0	0.26
sample 2	6.9	4.3	0.25
sample 3	13.7	3.3	0.27
sample 4	22.6	2.5	0.27

(Data from the Griffith experiment)

$$\sigma_c\sqrt{a} = \text{const}$$

Griffith's Energy Balance

- All energetic changes are caused by changes in crack size:

$$\frac{\partial W}{\partial a} = \frac{\partial U_e}{\partial a} + \frac{\partial U_i}{\partial a} + \frac{\partial U_k}{\partial a} + \frac{\partial U_\Gamma}{\partial a}$$

- For brittle materials and slow processes:

$$\Pi = U_e - W \Rightarrow -\frac{\partial \Pi}{\partial a} = \frac{\partial U_\Gamma}{\partial a} = 2\gamma_s$$

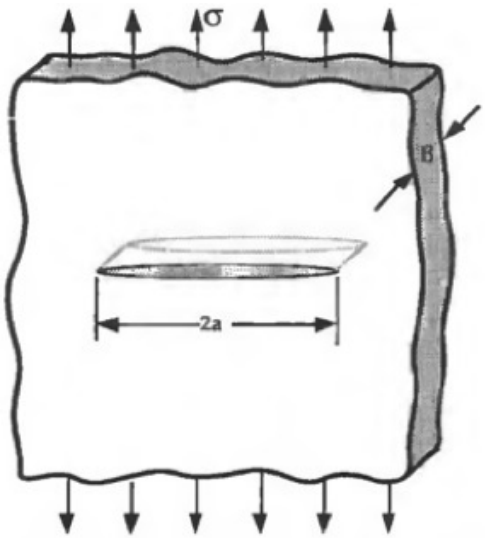
γ_s : Energy required to form a unit surface area

Griffith's Energy Balance

- Using Inglis's solution for an elliptical crack:

$$W = \pi a^2 B \frac{\sigma^2}{E} \qquad U_\Gamma = 4aB\gamma_s$$

$$\Rightarrow -\frac{\partial \Pi}{\partial a} = 2\pi a B \frac{\sigma^2}{E} \qquad \frac{\partial U_\Gamma}{\partial a} = 4B\gamma_s$$



- From energy balance:

$$\sigma_c = \sqrt{\frac{2E\gamma_s}{\pi a}}$$

Energy Balance for Ductile Materials

- Irwin, Orowan (1948):

$$\sigma_c = \sqrt{\frac{2E\left(\gamma_s + \gamma_p\right)}{\pi a}}$$

γ_s : Plastic work per unit surface area created

- Typically in metals, $\gamma_p \approx 1000\gamma_s$

- Not a material constant

Energy Release Rate: Irwin (1956)

$$G = -\frac{\partial \Pi}{\partial a}$$

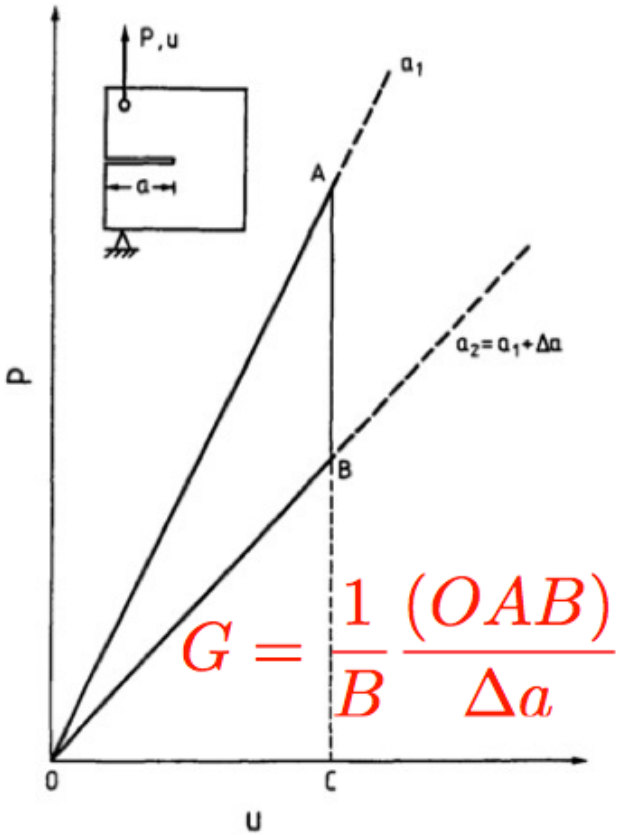
G : Energy released during fracture per created crack surface area

- Energy release rate failure criterion,

$$G \geq G_c = 2\left(\gamma_s + \gamma_p\right)$$

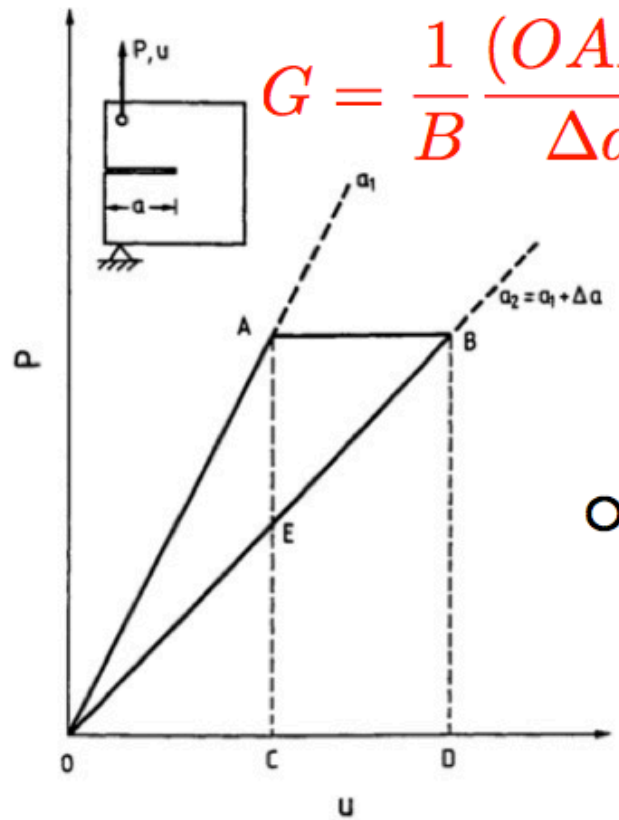
Energy Release Rate Measurement

Fixed grips



$$G = \frac{1}{B}\frac{(OAB)}{\Delta a}$$

Dead loads



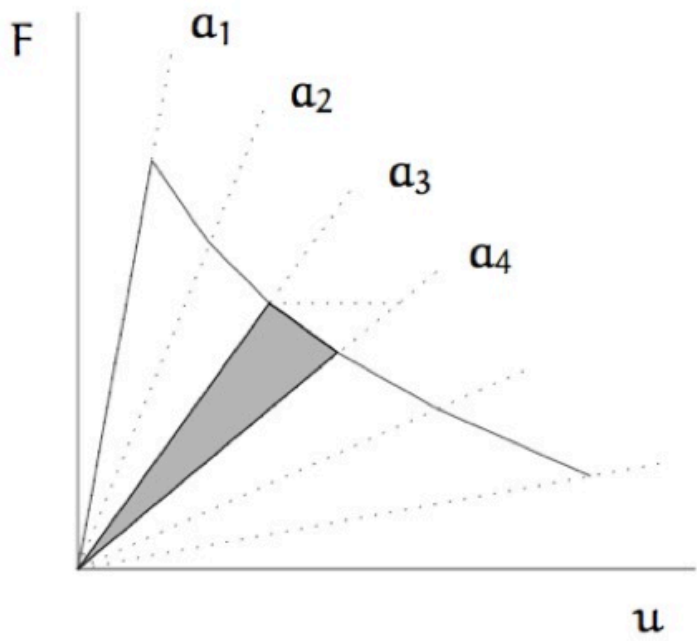
$$G = \frac{1}{B}\frac{(OAB)}{\Delta a}$$

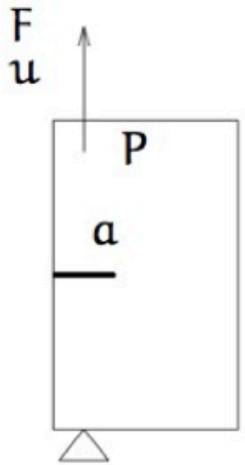
B: thickness

$$\Pi = U_e - W$$

OAB=ABCD-(OBD-OAC)

Energy Release Rate Measurement





$$G = \frac{1}{B}\frac{\text{shaded area}}{a_4 - a_3}$$

39

Crack Growth Resistance Curve

- Energy balance with plasticity:

$$-\frac{\partial \Pi}{\partial a} = \frac{\partial U_\Gamma}{\partial a} + \frac{\partial U_i}{\partial a}$$

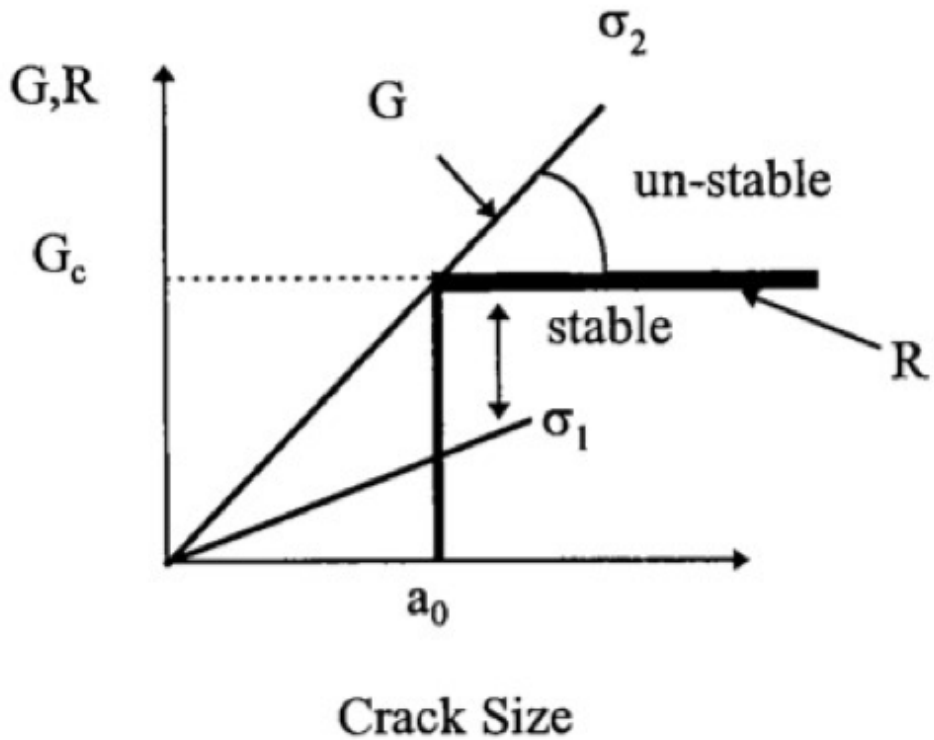
$$R \equiv \frac{\partial U_\Gamma}{\partial a} + \frac{\partial U_i}{\partial a} \qquad R: \text{Crack growth resistance}$$

- R increases with growing crack size in plastic materials

- Not a material constant

R-Curves: Brittle

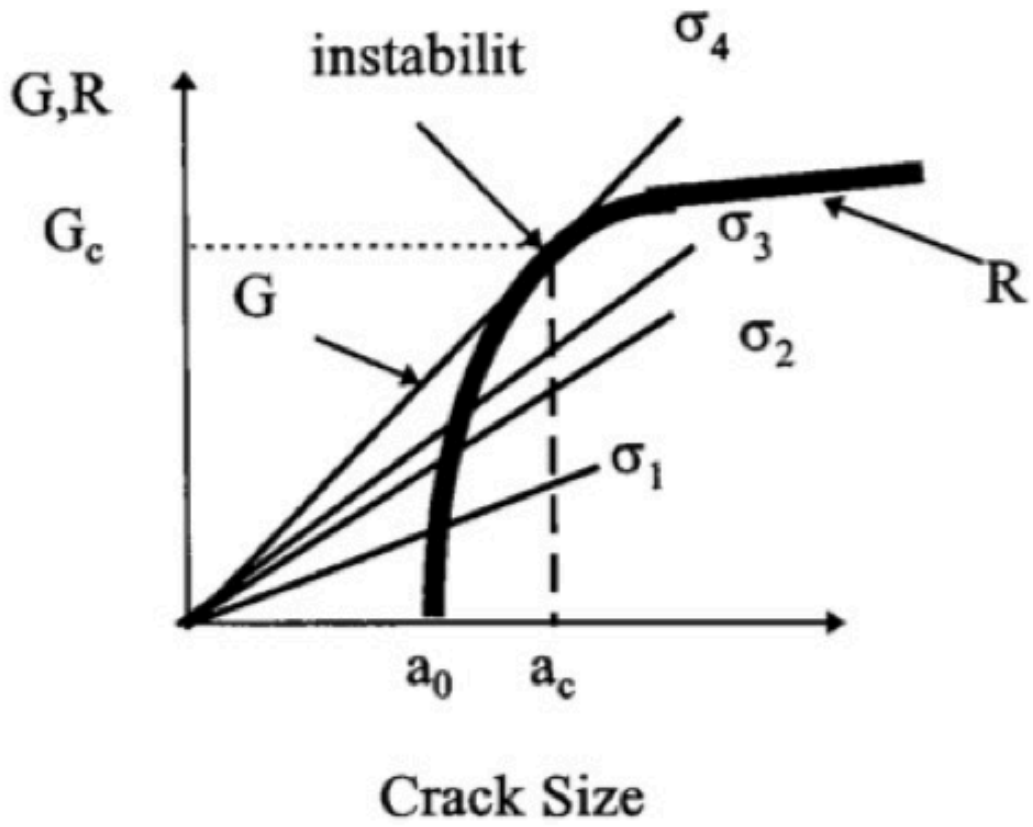
- Flat R-Curve: Brittle materials



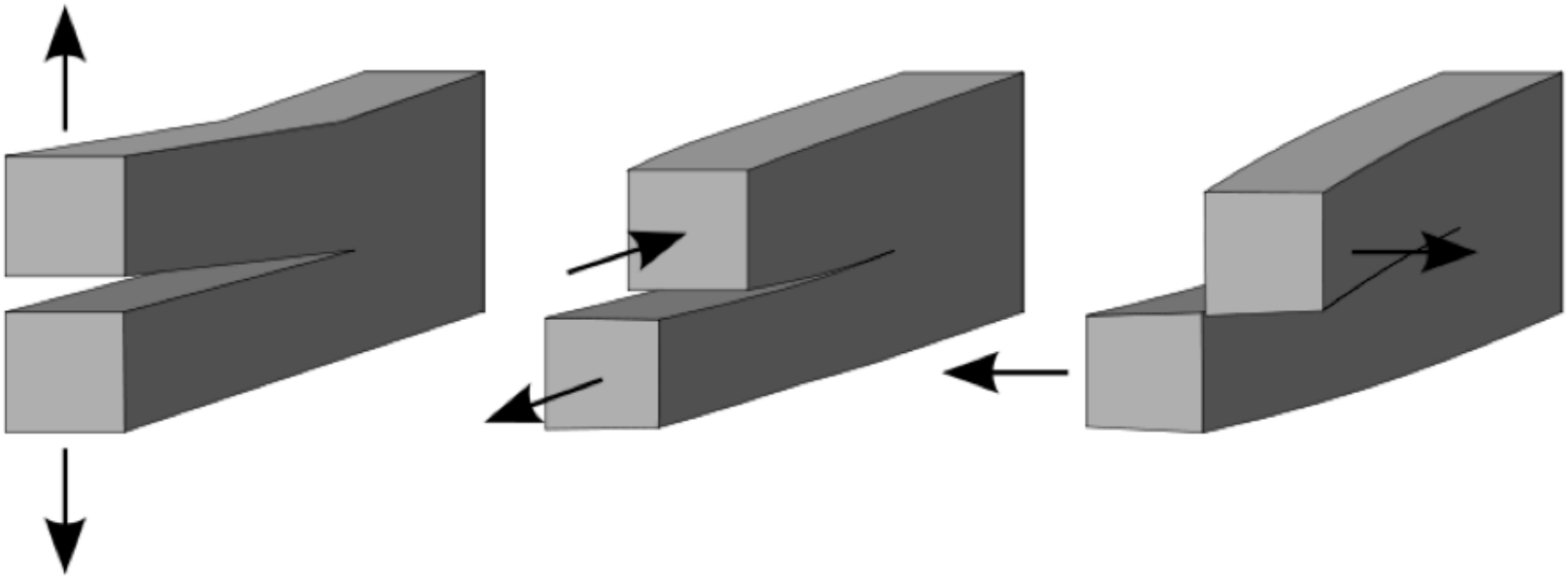
$$\frac{\partial G}{\partial a} \leq \frac{\partial R}{\partial a} \rightarrow \text{Stable crack growth}$$

R-Curves: Ductile

- Rising R-Curve: Ductile materials



Crack Modes



Mode I: Opening

Mode II: In-plane shear

Mode III: Out-of-plane shear

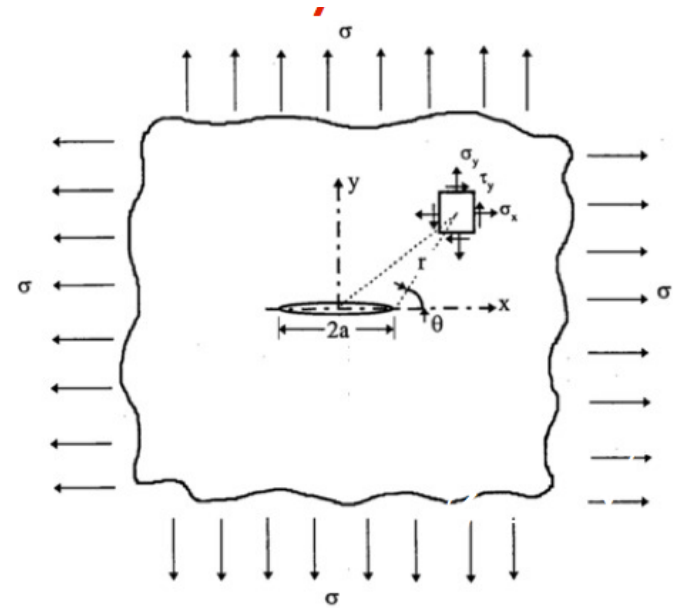
Crack Tip Stress Field: Mode I

- Westergaard (1937)

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

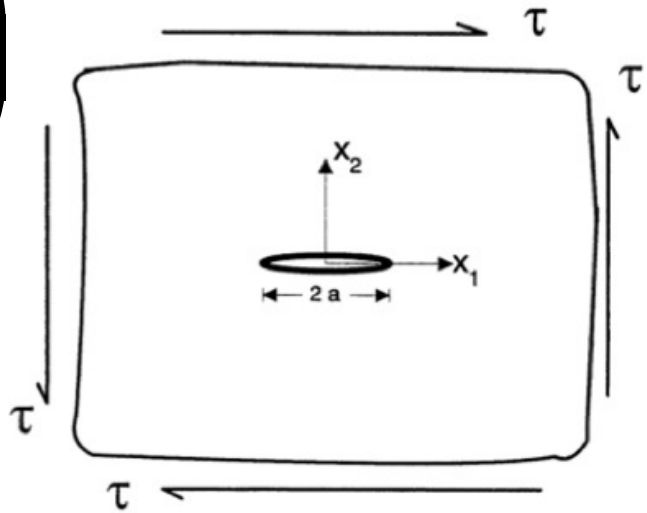


Crack Tip Stress Field: Mode II

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$



K_I, K_{II} : Stress intensity factor

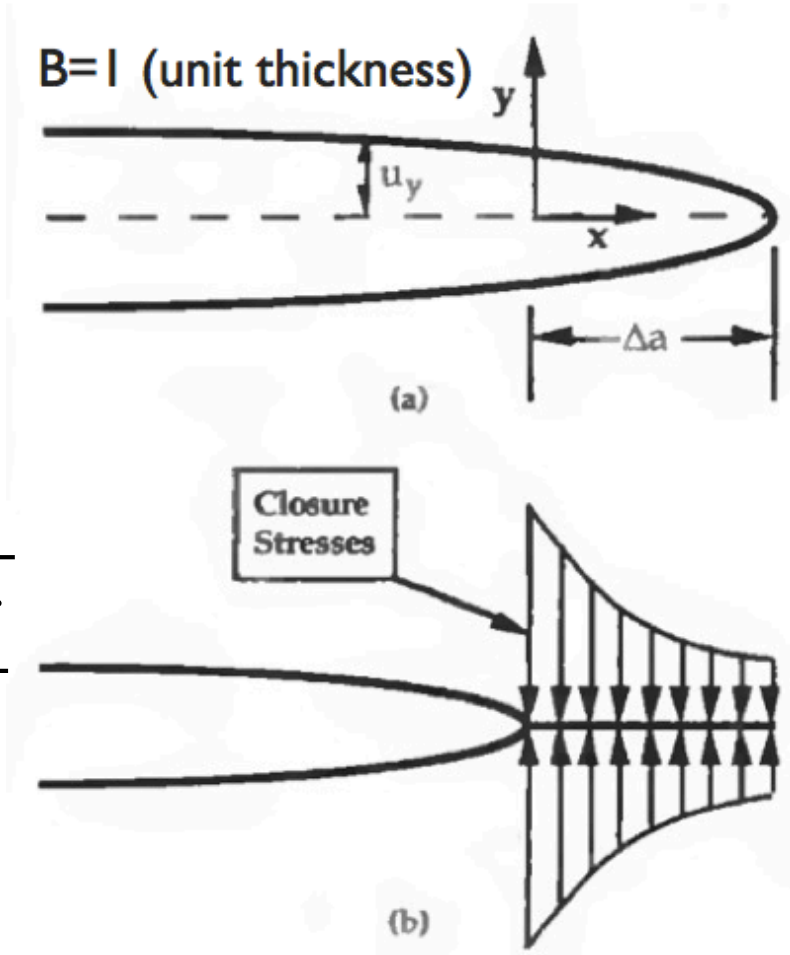
K-G relationship

- Work to required to open a crack, G, is the same as the work required to close a crack

$$\Delta W = \int_0^{\Delta a} \sigma_{yy} u_y \, dx$$

$$u_y = \frac{(\kappa + 1) K_I (a + \Delta a)}{2\mu} \sqrt{\frac{\Delta a - x}{2\pi}}$$

$$\sigma_{yy} = \frac{K_I(a)}{\sqrt{2\pi x}}$$



K-G relationship

Mode I

$$G_I = \begin{cases} \dfrac{K_I^2}{E} & \text{plane stress} \\ (1-v^2)\dfrac{K_I^2}{E} & \text{plane strain} \end{cases}$$

Mixed mode

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

$$E' = \begin{cases} \dfrac{E}{1-\nu^2} & \text{for plane strain} \\ E & \text{for plane stress} \end{cases}$$

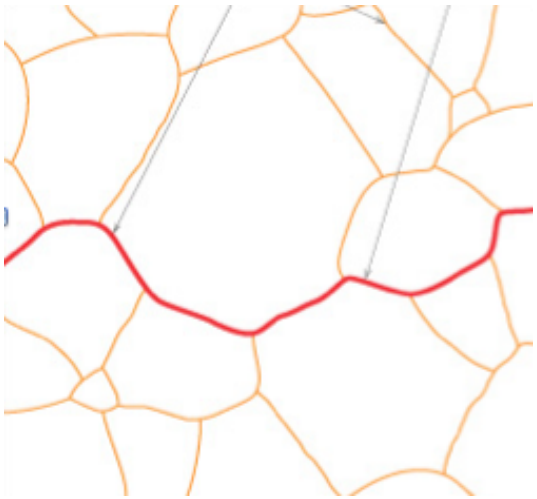
Varational Brittle Fracture: Francfort, Bourdin

- Formulate Griffith's energy balance as a minimum energy principle

$$\frac{\partial \Pi}{\partial a} + \frac{\partial U_\Gamma}{\partial a} = 0 \rightarrow \min_a \left(\Pi + U_\Gamma \right)$$

- Couple with mechanics

$$\min_{u,a} \int_\Omega \Pi(u,a) d\Omega + \int_a 2\gamma_s \, da$$



Phase Field Regularization

- Minimization over all possible crack surfaces is numerically challenging

- Phase Field approximation of the surface integral

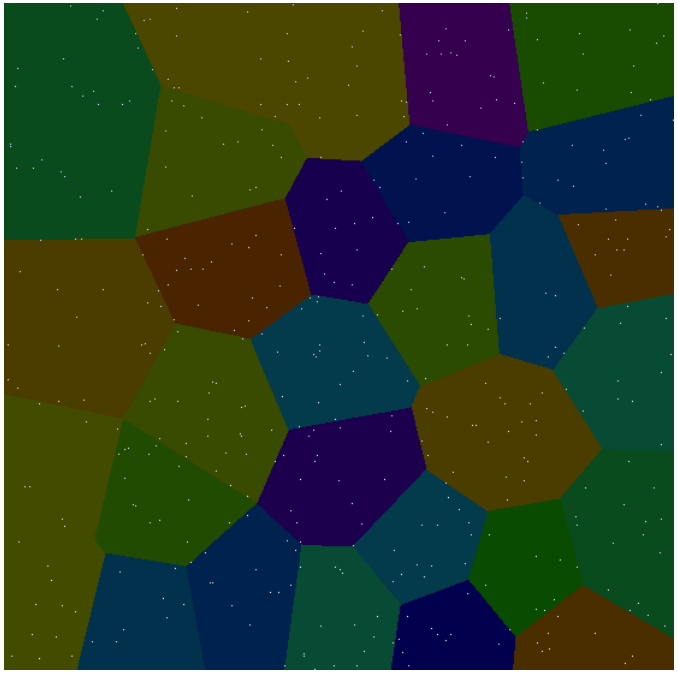
$$\min_{u,\varphi} \underbrace{\int_{\Omega} \varphi^2 \Pi(u) d\Omega}_{\text{Elastic energy}} + \underbrace{\int_{\Omega} 2\left(\gamma_s \ell \left|\nabla \varphi\right|^2 + \frac{\gamma_s}{\ell}(1-\varphi)^2\right) d\Omega}_{\text{Surface energy}}$$

- Starionary condition:

$$\nabla \bullet \varphi^2 \frac{\delta \Pi}{\delta \nabla u} = 0 \qquad 2\gamma_s \ell \Delta \varphi - \frac{\gamma_s}{\ell}(1-\varphi) - 2\varphi \Pi = 0$$

Examples

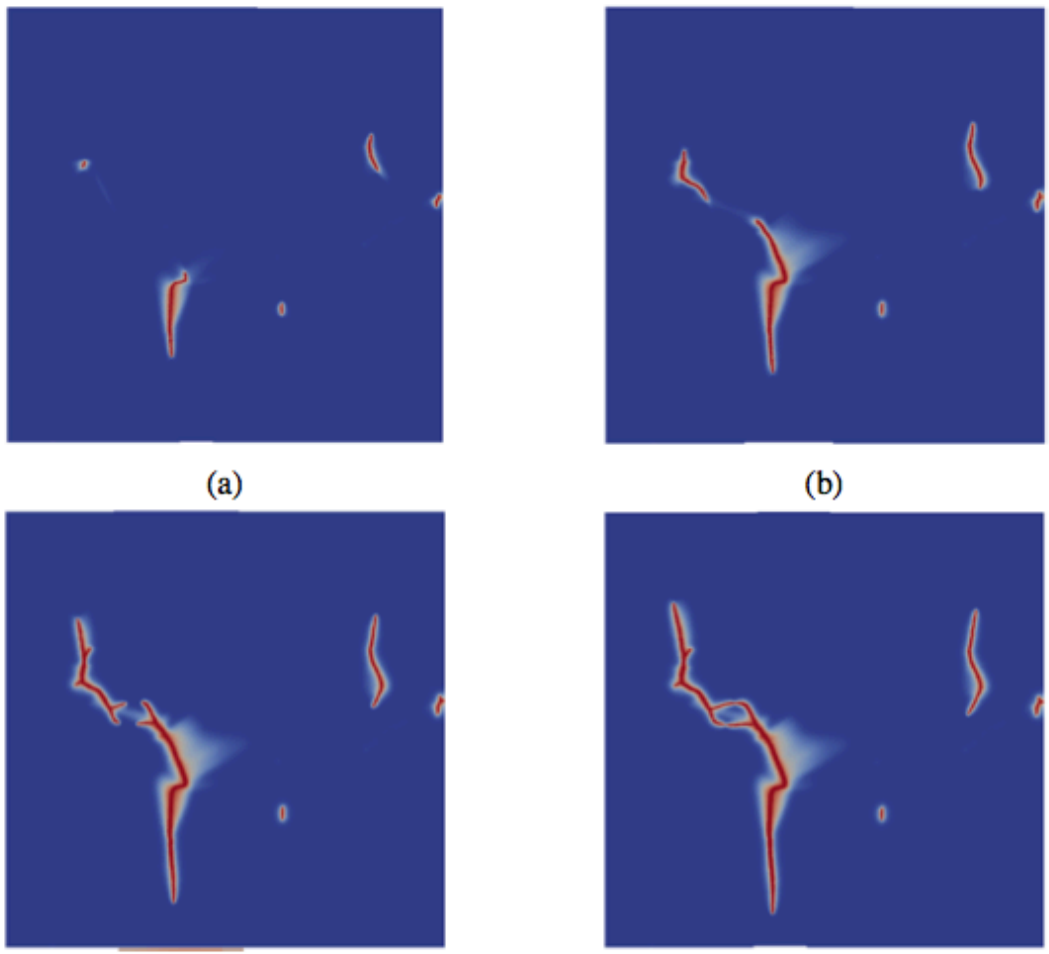
- PolyCrystalline fracture mechanics



Parameter	Unit	Value
C_{11}	GPa	168.0
C_{12}	GPa	121.4
C_{44}	GPa	28.34
$\dot{\gamma}_0$	s^{-1}	1e-3
n		20
g_0	MPa	31
g_∞	MPa	63
a		2.25
h_0	MPa	75
coplanar $h_{\alpha\beta}$		1
non-coplanar $h_{\alpha\beta}$		1.4
G_0	Jm^{-2}	1.0
l_0	µm	1.5
M	s^{-1}	0.01

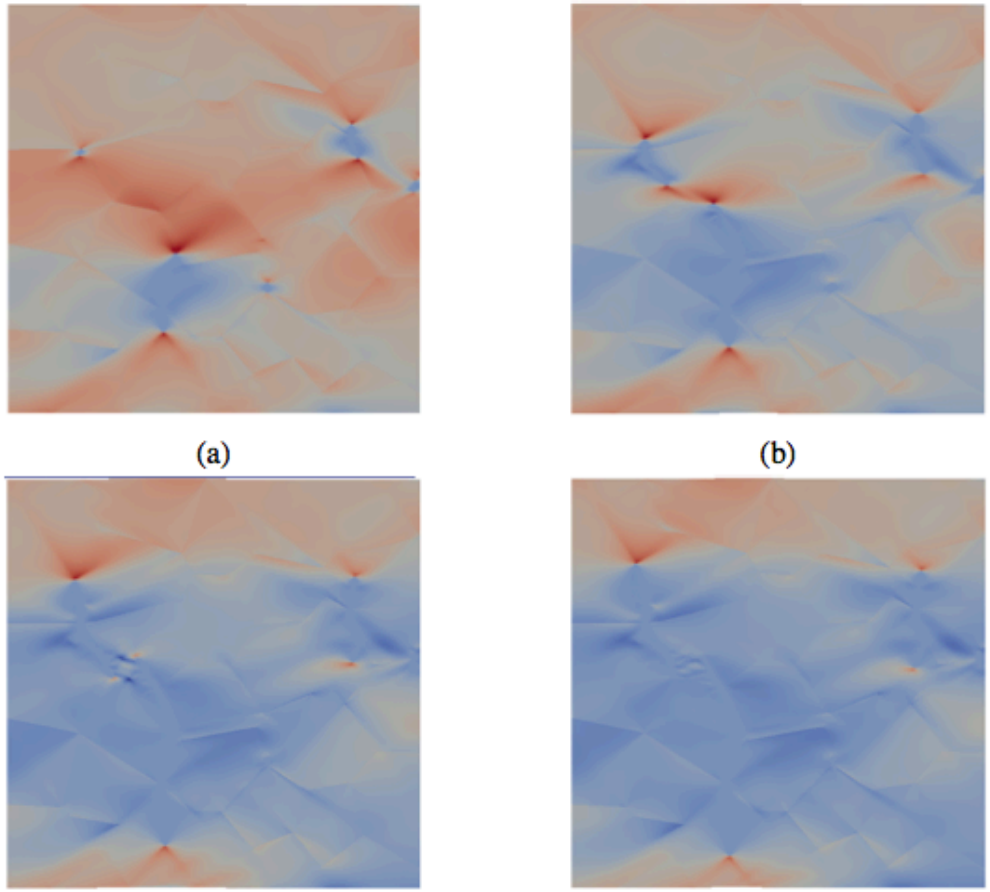
Examples

- Evolving crack patterns



Examples

- Elastic energy release and crack tip stress



(a) (b) (c) (d)

Summary

DAMASK - The Düsseldorf Advanced Material Simulation Kit is free software licensed under the terms of the GNU General Public License.

damask.mpie.de