The Finite Element Method: Introduction

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• Introduction & historical developments
• Theory: variational calculus, weak form
• The finite element method
• 1D structural mechanics example
• Element types
• Advanced topics
Introduction: Basic idea of FEM

• Basic idea of FEM: Simplify a complex geometry into simple shapes called finite elements

• Solve the simplified problem inside each finite element

• Obtain the solution to the complex problem by assembling the solution of each finite element
Finite element model

• Example: Structural mechanics

\[
[K] = \text{Stiffness matrix of the part} \\
\text{(Sum of all elements)}
\]
\[
\{U\} = \text{Components of the displacements} \\
\text{of the single nodes of the part}
\]
\[
\{F\} = \text{Components of the loads of} \\
\text{the single nodes of the part}
\]

\[
[K] \{U\} = \{F\}
\]

• Approach the original problem with mesh refinement
Historical developments

• 1950s: Turner, Boeing, stiffness method
• 1960s: Zienkiewicz, Clough, FEM technology development
• 1970s: FEM software development (ANSYS, NASTRAN, ABAQUS)
• 1990s: Introduction of generalized FEMS (XFEM, isogeometric FEM)
• 2000: FEM is the standard tool for structural analysis
Theory: Structural mechanics

- Deformation map
  \[ \chi : B_0 \rightarrow B \]

- Deformation gradient
  \[ F = \frac{\partial \chi}{\partial X} \]

- Constitutive relation
  \[ P = \frac{\delta W}{\delta F} = f(x,F,\dot{F},\xi...) \]

- Mechanical equilibrium
  \[ \min W \Rightarrow \nabla \cdot P = f \]
Theory: Weak form

- Test functions, $v$

\[ \int v : \nabla \cdot P \, dV = \int v : f \, dV \quad \forall v \quad \leftrightarrow \quad \nabla \cdot P = f \]

- Integration by parts

\[ \int \nabla v : P \, dV - \int v : P \cdot n \, dS = \int v : f \, dV \quad \forall v \]

Boundary condition
Finite element method

• Example 1D bar

• Discrete mesh

Element 1   Element 2

Node 1          Node 2          Node 3
FEM: Shape functions (i)

- Displacement field is approximated by shape functions

\[ u(x) = \sum_i N_i(x)u_i \]

- Displacement is linearly interpolated between values at nodes

\[ N_1(x) = \frac{x_2 - x}{x_2 - x_1} \]
\[ N_2(x) = \frac{x - x_1}{x_2 - x_1} \]
• Displacement is compatible across elements

\[ u^{(1)}(x) = \frac{x_2-x}{x_2-x_1} u_{1x} + \frac{x-x_1}{x_2-x_1} u_{2x} \]

\[ u^{(2)}(x) = \frac{x_3-x}{x_3-x_2} u_{2x} + \frac{x-x_2}{x_3-x_2} u_{3x} \]

• Properties:

\[ N_i(x_j) = \delta_{ij} \]

\[ \sum_i N_i(x) = 1 \]
FEM: Strain matrix

- Strain-displacement relation

\[
\varepsilon(x) = \frac{du}{dx}(x) = \sum_i \frac{dN_i}{dx}(x)u_i = \sum_i B_i u_i
\]

- For linear element

\[
\mathbf{N} = \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} = \begin{bmatrix} \frac{x_2 - x}{x_2 - x_1} & \frac{x - x_1}{x_2 - x_1} \end{bmatrix}
\]

\[
\mathbf{B} = \begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix}
\]

Strain is constant in a linear element
FEM: Discrete weak form

- **Galerkin**: Approximate test functions and displacement using shape functions

\[
\begin{align*}
  u(x) &= \sum_i N_i(x) u_i \\
  v(x) &= \sum_i N_i(x) v_i
\end{align*}
\]

- **Discrete weak form**

\[
\begin{align*}
  v_i^T \left( \int B_i^T E B_j \, dV \right) u_j &= v_i^T \left( \int N_i^T f \, dV \right) \quad \forall v_i
\end{align*}
\]
FEM: Stiffness matrix

- For 1D element $dV^e = A^e dx^e$

$$\Rightarrow \sum_e \left[ \left( \int_{x_1^e}^{x_2^e} B^T EBA dx^e \right) u^e \right] = \sum_e \left[ \int_{x_1^e}^{x_2^e} N^T fA dx^e \right]$$

- Where, stiffness matrix, $K^e$, is given by

$$K^e = \int_{x_1^e}^{x_2^e} B^T EBA dx^e = \frac{A^e E}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
FEM: Global assembly

• Global stiffness matrix is assembled by combining element stiffness matrices at the connecting nodes

\[
K = \sum_{e} K^e = \begin{bmatrix}
\frac{EA^1}{L^1} & -\frac{EA^1}{L^1} & 0 \\
-\frac{EA^1}{L^1} & \frac{EA^1}{L^1} & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & \frac{EA^2}{L^2} & -\frac{EA^2}{L^2} \\
0 & -\frac{EA^2}{L^2} & \frac{EA^2}{L^2}
\end{bmatrix}
\]

Element 1  Element 2

Node 1  Node 2  Node 3
FEM: Boundary conditions

- Essential boundary conditions: Enforced in test function space directly

\[ u(0) = 0 \implies v(0) = 0 \]

\[ K_{BC} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \]

- Natural boundary conditions: Enforced in the weak form

\[ P \cdot n(0) = f_n \implies \int v : P \cdot n \, dS_n = \int v : f_n \, dS_n \]
• Global system: \( K_{BC}u = \tilde{f} + \tilde{f}_n \)

\[
\begin{bmatrix}
\frac{EA^1}{L^1} + \frac{EA^2}{L^2} & -\frac{EA^2}{L^2} \\
-\frac{EA^2}{L^2} & \frac{EA^2}{L^2}
\end{bmatrix}
\begin{Bmatrix}
\begin{align*}
u_2 \\
u_3
\end{align*}
\end{Bmatrix} =
\begin{Bmatrix}
0 \\
F
\end{Bmatrix}
\]

• Solve for displacements
Finite element zoo

First order

Line
Tetrahedral

Second order

Line
Tetrahedral

Triangle/Quadrilateral (tri/quad)

Hexahedral (hex)

10-noded
Advanced topics: Isoparametric elements

- Define “standard element” in isoparametric domain $\xi = [-1,1]^D$

- Use shape functions to map geometry

\[
x(\xi) = \sum_i N_i(\xi) x_i
\]

Same shape functions for all elements

- Stiffness matrix

\[
K^e = \int_{-1}^{1} (JB)^T E (JB) \det(J) d\xi
\]

\[
J = \frac{\partial \xi}{\partial x}
\]
Advanced topics: Numerical integration

• Gauss quadrature

\[ K^e = \int_{-1}^{1} f(\xi) \, d\xi \approx \sum_i w_i f(\xi_i) \]

• Reduced integration: Usage of sub-optimal quadrature rule will result in zero-energy modes (eg. Hourglassing)
Thank you for your attention!