The Finite Element Method: Indroduction

P. Shanthraj, D. Raabe

Max-Planck-Institut für Eisenforschung GmbH

Department of Microstructure Physics and Alloy Design

p.shanthraj@mpie.de

Outline

- Introduction & historical developments
- Theory: variational calculus, weak form
- The finite element method
- 1D structural mechanics example
- Element types
- Advanced topics

Introduction: Basic idea of FEM

- Basic idea of FEM: Simplify a complex geometry into simple shapes called finite elements

- Solve the simplified problem inside each finite element

- Obtain the solution to the complex problem by assembling the solution of each finite element

Finite element model

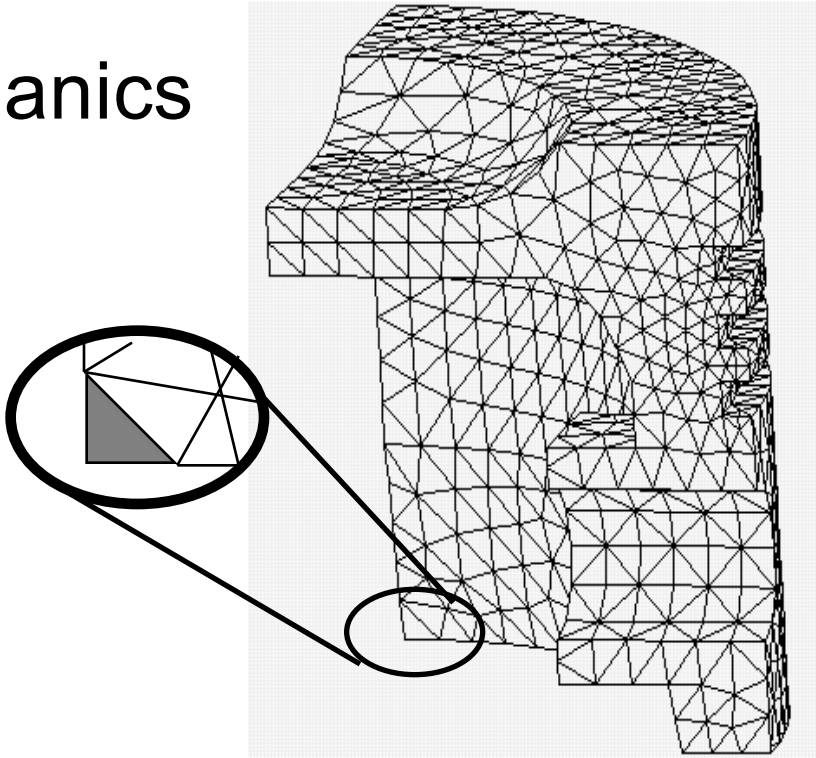
- Example: Structural mechanics

  [K] = Stiffness matrix of the part
  (Sum of all elements)

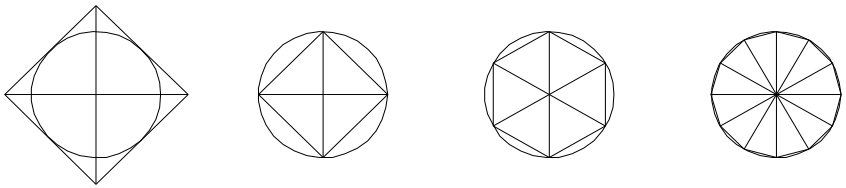
  {U} = Components of the displacements of the single nodes of the part

  {F} = Components of the loads of the single nodes of the part

  [K] *{U} = {F}



- Approach the original problem with mesh refinement



Historical developments

- 1950s: Turner, Boeing, stiffness method
- 1960s: Zienkiewicz, Clough, FEM technology development
- 1970s: FEM software development (ANSYS, NASTRAN, ABAQUS)
- 1990s: Introduction of generalized FEMS (XFEM, isogeometric FEM)
- 2000: FEM is the standard tool for structural analysis

Theory: Structural mechanics

- Deformation map

$$\chi : B_0 \rightarrow B$$

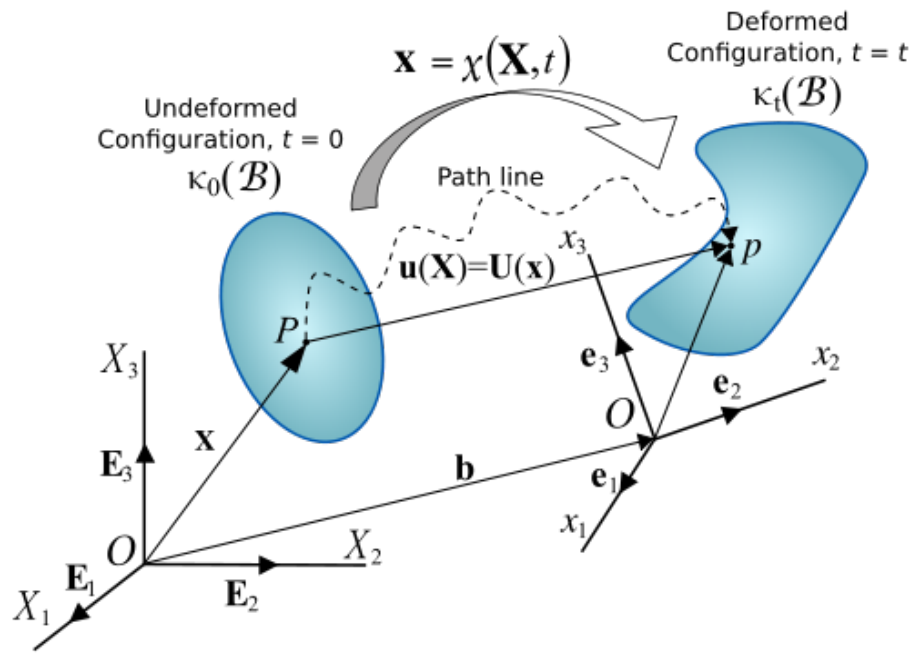
- Deformation gradient

$$F = \frac{\partial \chi}{\partial X}$$

- Constitutive relation

$$P = \frac{\delta W}{\delta F} = f\left(x, F, \dot{F}, \xi \ldots\right)$$

- Mechanical equilibrium $\min_{\chi} W \Rightarrow \nabla \cdot P = f$



Theory: Weak form

- Test functions, *v*

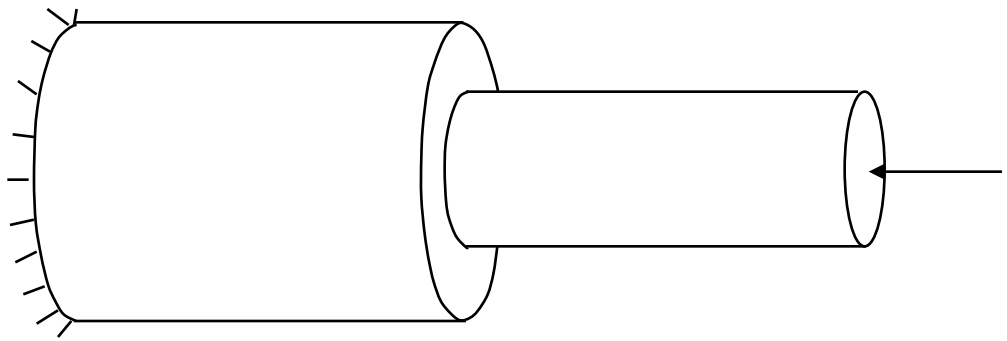
$$\int v : \nabla \cdot P \, dV = \int v : f \, dV \qquad \forall v \quad \Leftrightarrow \quad \nabla \cdot P = f$$

- Integration by parts

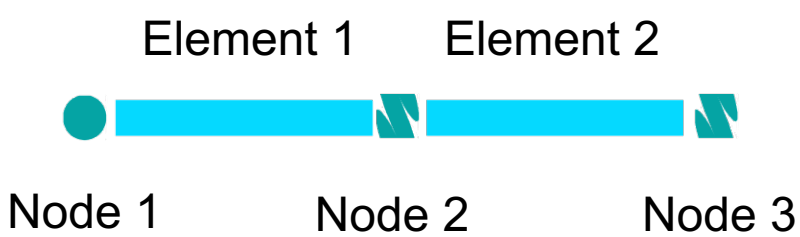
$$\int \nabla v : P \, dV - \underbrace{\int v : P \cdot n \, dS}_{\text{Boundary condition}} = \int v : f \, dV \qquad \forall v$$

Finite element method

- Example 1D bar



- Discrete mesh



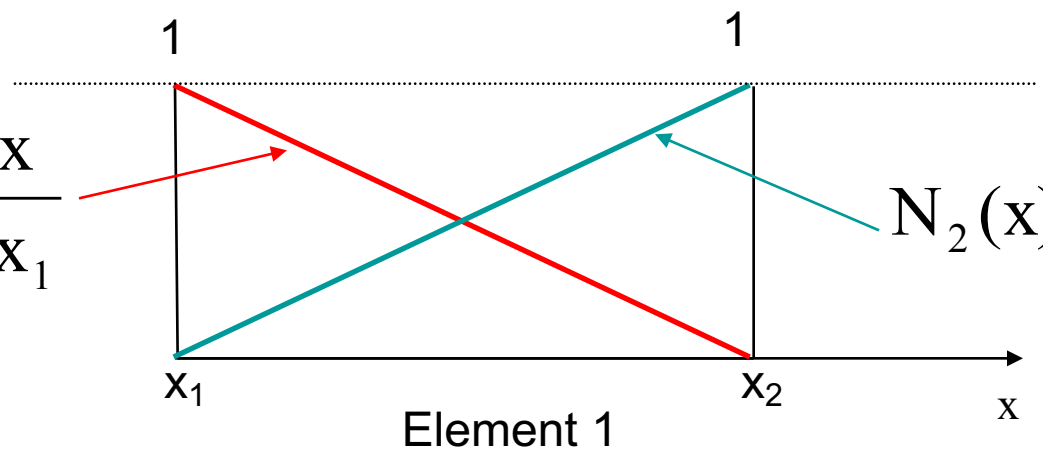
FEM: Shape functions (i)

- Displacement field is approximated by shape functions

$$u(x)=\sum_i N_i(x)u_i$$

- Displacement is linearly interpolated between values at nodes

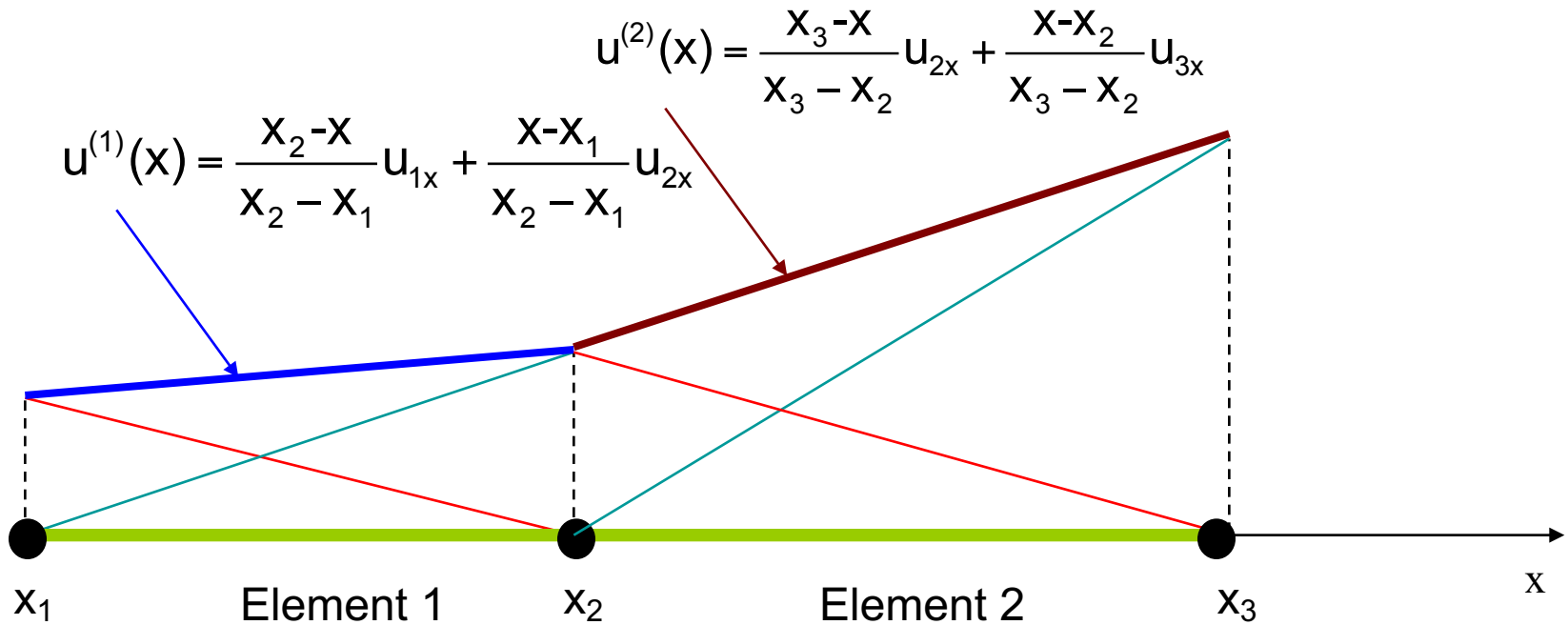
$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

FEM: Shape functions (ii)

- Displacement is compatible across elements



- Properties: $N_i(x_j) = \delta_{ij}$

$$\sum_i N_i(x) = 1$$

FEM: Strain matrix

- Strain-displacement relation

$$\varepsilon(x)=\frac{du}{dx}(x)=\sum_i \frac{dN_i}{dx}(x)u_i = \sum_i B_i u_i$$

- For linear element

$$\underline{N} = \left[N_1(x) \quad N_2(x)\right] = \left[\frac{x_2 - x}{x_2 - x_1} \quad \frac{x - x_1}{x_2 - x_1}\right]$$

$$\underline{B} = \left[\frac{-1}{x_2 - x_1} \quad \frac{1}{x_2 - x_1}\right] = \frac{1}{x_2 - x_1}\left[-1 \quad 1\right]$$

Strain is constant in a linear element

FEM: Discrete weak form

- Galerkin: Approximate test functions and displacement using shape functions

$$u(x) = \sum_i N_i(x) u_i \qquad v(x) = \sum_i N_i(x) v_i$$

- Discrete weak form

$$v_i^T \left(\int B_i^T E B_j \, dV \right) u_j = v_i^T \left(\int N_i^T f \, dV \right) \qquad \forall v_i$$

FEM: Stiffness matrix

- For 1D element $dV^e = A^e dx^e$

$$\Rightarrow \sum_e \left[\left(\int_{x_1^e}^{x_2^e} B^T EBA\, dx^e \right) u^e \right] = \sum_e \left[\int_{x_1^e}^{x_2^e} N^T fA\, dx^e \right]$$

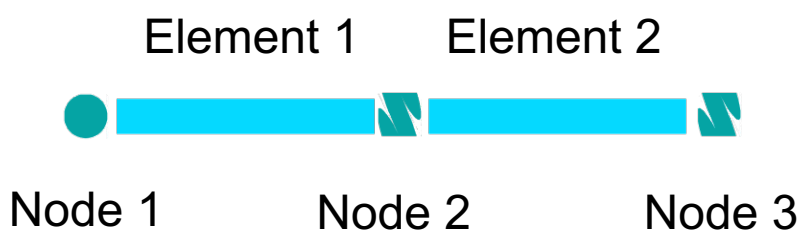
- Where, stiffness matrix, K^e, is given by

$$K^e = \int_{x_1^e}^{x_2^e} B^T EBA\, dx^e = \frac{A^e E}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

FEM: Global assembly

- Global stiffness matrix is assembled by combining element stiffness matrices at the connecting nodes

$$K = \sum_e K^e = \begin{bmatrix} \frac{EA^1}{L^1} & -\frac{EA^1}{L^1} & 0 \\ -\frac{EA^1}{L^1} & \frac{EA^1}{L^1} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{EA^2}{L^2} & -\frac{EA^2}{L^2} \\ 0 & -\frac{EA^2}{L^2} & \frac{EA^2}{L^2} \end{bmatrix}$$



FEM: Boundary conditions

- Essential boundary conditions: Enforced in test function space directly

$$u(0) = 0 \Rightarrow v(0) = 0 \qquad K_{BC} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

- Natural boundary conditions: Enforced in the weak form

$$P \cdot n(0) = f_n \Rightarrow \int v : P \cdot n \, dS_n = \int v : f_n \, dS_n$$

FEM: Solution

- Global system: $K_{BC}u = \tilde{f} + \tilde{f}_n$

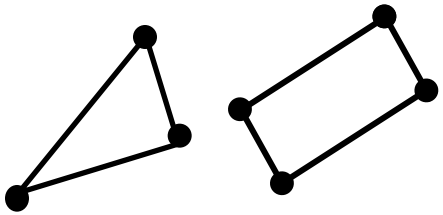
$$\begin{bmatrix} \frac{EA^1}{L^1} + \frac{EA^2}{L^2} & -\frac{EA^2}{L^2} \\ -\frac{EA^2}{L^2} & \frac{EA^2}{L^2} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

- Solve for displacements

Finite element zoo

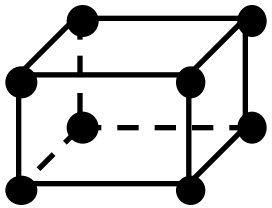
First order

Line
Tetrahedral



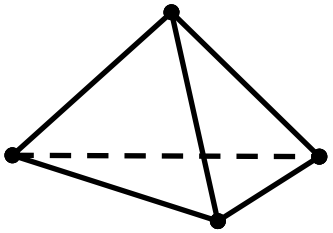
Triangle/Quadrilateral

(tri/quad)



Hexahedral

(hex)



(tet)

Second order

Line
Tetrahedral

Triangle/Quadrilateral

Hexahedral

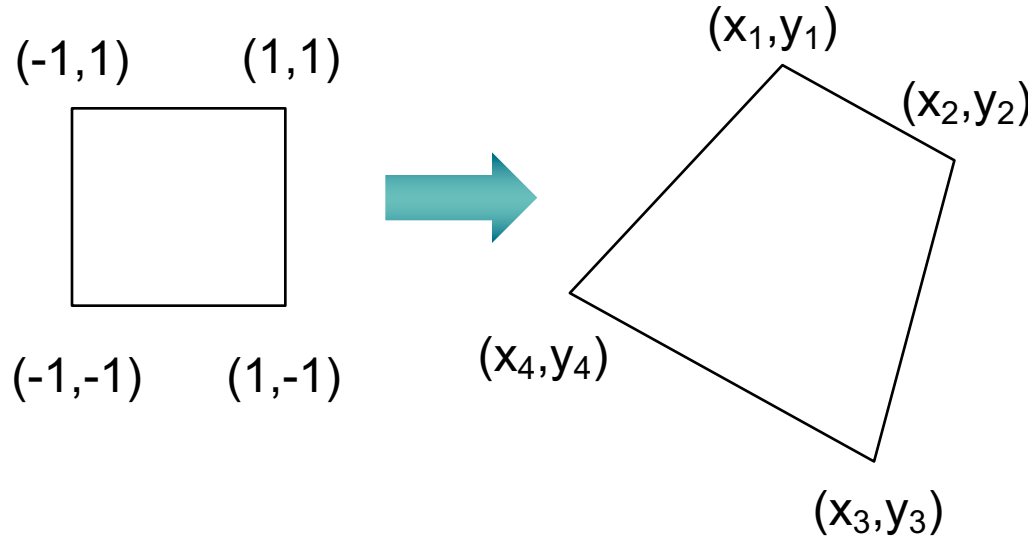
10-noded

Advanced topics: Isoparametric elements

- Define "standard element" in isoparametric domain $\xi = [-1,1]^D$
- Use shape functions to map geometry

$$x(\xi) = \sum_i N_i(\xi) x_i$$

Same shape functions for all elements



- Stiffness matrix

$$K^e = \int_{-1}^{1} (JB)^T E (JB) \det(J) \, d\xi \qquad J = \frac{\partial \xi}{\partial x}$$

Advanced topics: Numerical integration

- Gauss quadrature

$$K^e = \int_{-1}^{1} f(\xi)\, d\xi \approx \sum_i w_i f(\xi_i)$$

- Reduced integration: Usage of sub-optimal quadrature rule will result in zero-energy modes (eg. Hourglassing)

Thank you for your attention!